

The  $\mathcal{L}$  operator is the Laplace transform,  $\mathcal{L}[f(t)] = \tilde{f}(s) = \int_0^\infty dt e^{-st} f(t)$ . The  $\star$  operator is convolution,  $f \star g(t) = \int_0^t dt' f(t-t')g(t')$ .

1. Standard Laplace transform proofs.
  - (a) Prove that  $\mathcal{L}[f \star g(t)] = \tilde{f}(s)\tilde{g}(s)$ .
  - (b) Prove that  $\mathcal{L}[\dot{f}(t)] = s\tilde{f}(s) - f(0)$ .
2. Whence comes that  $2\pi$ ? Suppose we consider eigenfunctions of the time derivative operator that are periodic on the interval  $t = -T/2$  to  $T/2$ . The eigenfunction with eigenvalue  $i\omega$  is defined  $A_\omega \phi_\omega(t)$ , where  $A_\omega$  is a normalization constant and  $\phi_\omega(t)$  is the eigenfunction at time  $t$ . The phase of the eigenfunctions are fixed by requiring that  $\phi_\omega(0) = 1$ .

- (a) Provide the function  $\phi_\omega(t)$ .
- (b) The periodicity requirement is that  $\phi_\omega(-T/2) = \phi_\omega(+T/2)$ . What values of  $\omega$  are permitted?
- (c) What is the spacing  $\Delta\omega$  between permitted values? This quantization of permitted frequencies is the same phenomenon as the quantization of energy levels for particles in confining potentials, for example a particle in a box or an electron confined by the positive charge of a nucleus.
- (d) The dot product for discrete vectors becomes an inner product for continuous functions. The inner product of functions  $f(t)$  and  $g(t)$  is

$$\langle f|g \rangle = \int_{-T/2}^{T/2} dt f^*(t)g(t),$$

where  $f^*(t)$  is the complex conjugate of  $f(t)$ . What is  $\langle \phi_\omega | \phi_\omega \rangle$ ?

- (e) What is  $\langle \phi_\omega | \phi_{\omega'} \rangle$  for  $\omega \neq \omega'$ ?
- (f) For normalization, we require that

$$\sum_{\omega'} \Delta\omega \langle A_{\omega'} \phi_{\omega'} | A_\omega \phi_\omega \rangle = 1.$$

Note that only the term with  $\omega = \omega'$  contributes to the integral. What is the resulting value for the square magnitude of the normalization constant,  $|A_\omega^* A_\omega|$ ? How does the normalization constant depend on the frequency  $\omega$ ?

- (g) In the limit  $T \rightarrow \infty$ ,  $\Delta\omega \rightarrow 0$ , the eigenfunctions are written  $A(\omega)\phi(\omega, t)$ . What is  $A(\omega)\phi(\omega, t)$ ? For historical reasons (“it seemed like a good idea at the time”), many fields (including ours) effectively use  $\phi(\omega t)$  as the eigenfunction and interpret  $|A(\omega)|^2$  as a normalization for the frequency-domain integral.
- (h) Suppose a function  $f(t)$  is a superposition of eigenfunctions,  $f(t) = \int_{-\infty}^\infty \hat{f}(\omega)\phi(\omega, t)$ . How do we extract  $\hat{f}(\omega)$ ?