The Laplace transform operator is $\mathscr{L}$, defined as $\mathscr{L}[f(t)] = \tilde{f}(s) = \int_0^\infty dt e^{-st} f(t)$. The convolution operator is $\star$, defined as $f \star g(t) = \int_0^t dt' f(t-t')g(t')$. The real part of a complex variable $z = x + iy$ is denoted $\Re(z) = x$.

1. Suppose $f(t) = 1/(t^2+2)$.
   (a) Evaluate $f(2)$.
   (b) Evaluate $e^{[3\frac{d}{dt}f(t)]}$ at $t = 4$
   (c) Evaluate $(e^{3\frac{d}{dt}})f(t)$ at $t = 4$.
2. Suppose $f(t) = \sin(\omega t)$, with $f(t) = 0$ for $t < 0$. Let $g(t) = f(t-a)$, with $g(t) = 0$ for $t < a$.
   (a) Provide $\mathscr{L}[f(t)]$.
   (b) Provide $\mathscr{L}[g(t)]$.
   (c) Simplify $(e^{\frac{\pi}{2}\frac{d}{dt}})\sin(\omega t)$.
3. Let $x(t)$ be a time response. Moment n of the response is defined as

$$\langle t^n \rangle = \frac{\int_0^\infty dt\, t^n x(t)}{\int_0^\infty dt\, x(t)}.$$

   (a) Prove that $(-d/ds)^2 \ln \tilde{x}(s)|_{s=0} = \langle t^2 \rangle - \langle t \rangle^2$.
   (b) Provide $(-d/ds)^3 \ln \tilde{x}(s)|_{s=0}$ in terms of the moments.
4. The MAPK signaling cascade usually has three levels denoted $k \in \{1,2,3\}$. The activation state of level k at time t is denoted $x_k(t)$. When activation is weak, a linear model may be appropriate:

$$\begin{aligned} \dot{x}_1(t) &= b\beta(t) - \alpha x_1(t) \\ \dot{x}_2(t) &= b x_1(t) - \alpha x_2(t) \\ \dot{x}_3(t) &= b x_2(t) - \alpha x_3(t). \end{aligned}$$

   Consider an exponentially decaying input, $\beta(t) = \beta_0 k e^{-kt}$. For $t < 0$, $\beta(t) = 0$, and $x_k(0) = \dot{x}_k(0) = 0$ for $k \in 1 \ldots 3$. Provide all results in terms of model parameters $\{\beta_0, k, \omega, b, \alpha\}$, as well as s or t as appropriate.
   (a) Provide $\tilde{\beta}(s)$, $\tilde{x}_1(s)$, $\tilde{x}_2(s)$, and $\tilde{x}_3(s)$.
   (b) Provide the gain, $\int_0^\infty dt\, x_3(t) / \int_0^\infty dt\, \beta(t)$.
   (c) Provide the mean time of the response, $\langle t \rangle = \int_0^\infty dt\, t x_3(t) / \int_0^\infty dt\, x_3(t)$, which will include a contribution from the input $\beta(t)$.

(d) Provide the square width of the response,

$$\sigma^2 = \frac{\int_0^\infty dt\, t^2 x_3(t)}{\int_0^\infty dt\, x_3(t)} - \left[\frac{\int_0^\infty dt\, t\, x_3(t)}{\int_0^\infty dt\, x_3(t)}\right]^2.$$

5. Now consider a cascade with n steps with the same form as above and with output $x_n(t)$. The input is a δ-function, $\beta(t) = \beta_0 \delta(t)$, and the cascade off at time 0.

   (a) Provide $\tilde{x}_n(s)$, the gain, the mean time $\langle t \rangle$, and the square width σ^2 of the output.

   (b) We can think about the sharpness of a response as $\langle t \rangle / \sigma$, where the duration σ is $\sqrt{\sigma^2}$. How does the sharpness of the output depend on the number of elements n in the cascade?