Abstract

The purpose of this handout is to summarize what you need to know to solve the contour integration problems you will see in SBE 3. For the homeworks, quizzes, and tests you should only need the “Primary Formulas” listed in this handout. We will use these formula to work a few example problems. In addition I would highly recommend checking out the videos listed at the end of this handout. They were very helpful for me when I took this class.

Definition of notation + terms

$z$ - a complex variable consisting of real part $x$ and imaginary part $iy$.

$f(z)$ - Some function mapping complex number $z$ to another complex number.

$\Gamma$ - A simple closed curve (ie one that does not self-intersect).

Analytic - A function $f(z)$ is said to be analytic on a set (ie in a region) $G$ if it has a derivative at every point in $G$.

Primary Formulas

Residue Theorem

If $f(z)$ is analytic inside and on $\Gamma$ except at points $z_1, z_2, \ldots, z_n$ inside $\Gamma$. Then:

$$\int_{\Gamma} f(z)dz = \pm 2\pi i \sum_{j=1}^{n} \text{Res}(f; z_j)$$

In words: The contour integral of $f(z)$ over the curve $\Gamma$ is equal to $\pm 2\pi i$ multiplied by the sum of the residues of $f(z)$. When $\Gamma$ is positively oriented (counter-clockwise in the complex plane see figure 3 for an example) then the $\pm$ will be +. If $\Gamma$ is negatively oriented then it will be −. The residue of $f(z)$ at $z_0$ is defined as:

$$\text{Res}(f; z_0) = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)]$$

where $m$ is the order of the pole. Note that for simple poles (of order 1) the above equation simplifies considerably:

$$\text{Res}(f; z_0) = \lim_{z \to z_0} [(z - z_0)f(z)]$$

Jordan’s Lemma

This is a helpful theorem when we have to evaluate a contour integral along an infinite open half-circle as opposed to a closed contour (see figures 1 and 2). If $t > 0$ and $P/Q$ is the quotient of two polynomials such that:

$$\text{degree}(Q) \geq 1 + \text{degree}(P)$$

1 A figure-eight is not a simple closed curve as it intersects itself. Circles and polygons are simple closed.

2 As an example the function $f(z) = 1/(z - z_0)$ is analytic at every point in the complex plane except at $z = z_0$.

3 For example $f(z) = 1/(z - z_0)^n$ has a pole of order $n$ ($\forall n > 0$).
then
\[
\lim_{r \to +\infty} \int_{C_r^+} e^{izt} \frac{P(z)}{Q(z)} \, dz = 0 \quad (3)
\]
where \( C_r^+ \) is an open half circle in the upper half-plane (see figure 1). A more useful form of Jordan’s Lemma for some purposes will be when we are considering the integral in the left-half plane (figure 2). We can do a change of variables to the above equation such that
\[
s = iz \text{ and we get (for } t > 0): \lim_{r \to +\infty} \int_{C_r^-} e^{st} \frac{P'(s)}{Q'(s)} \, ds = 0 \quad (3')
\]
Where \( P'(s) \) and \( Q'(s) \) are \( P(\frac{s}{z}) \) and \( Q(\frac{s}{z}) \), respectively. \( C_r^- \) is an open half circle in the left half-plane. A similar change of variables can be done when considering the lower half-plane \( (s = -z, t < 0) \) or the right half-plane \( (s = -iz, t < 0) \).

**Worked Examples**

**Problem 1 - A line integral in \( \mathbb{R}^2 \)**

As a refresher on how to do line integrals we briefly go through one. Contour integrals in the complex plane are in many ways similar to line integrals in 2D.

**Problem Statement.** Given vector field: \( \vec{f}(x, y) = 5x^2yi + 3xyj \) evaluate the line integral \( \int_C \vec{f} \cdot d\vec{r} \), where \( C \) is given by the path of the parabola \( \vec{r} = 5t^2i + tj \) for \( 0 < t < 1 \).
Solution. First we plug the path into our vector field:

$$\vec{f}(r(t)) = 125t^3i + 15t^3j$$

Then we solve for $d\vec{r}$:

$$d\vec{r} = dt(10ti + j)$$

Now we compute the dot product inside the integral and plug in our limits for $t$:

$$\int_C \vec{f} \cdot d\vec{r} = \int_0^1 (1250t^4 + 15t^3)dt$$

We've simplified to a standard integral so now we solve:

$$\int_0^1 (1250t^4 + 15t^3)dt = \frac{621}{2}$$

Problem 2 - A contour integral in $\mathbb{C}$

Problem Statement. Solve for the contour integral:

$$\int_{C_r} f(z)dz$$

where $f(z) = (z - z_0)^n$. Assume that $n$ is an integer ($-\infty < n < +\infty$) and $C_r$ is the circle $|z - z_0| = r$. We are transversing the contour in the positive direction (counterclockwise). \[\text{See figure 3.}\]

Solution. Note that because the contour is a circle it makes more sense to parameterize $z$ in polar coordinates. We parameterize the contour $C_r$ as $z(t) = z_0 + re^{it}, 0 \leq t \leq 2\pi$. Now we can plug in:

$$f(z(t)) = (z_0 + re^{it} - z_0)^n = r^n e^{int}$$

We solve for $dz$:

$$dz = ire^{it}dt$$

Plugging into our integral we get:

$$\int_{C_r} (z - z_0)^n dz = \int_0^{2\pi} (r^n e^{int})(ire^{it})dt$$

Noting that our radius does not change around the contour we have:

$$= ir^{n+1} \int_0^{2\pi} e^{i(n+1)t}dt$$

\[\text{4Note that we’ve used the identities: } i \cdot j = 0 \text{ and } i \cdot i = 1. \text{ In the complex plane these identities are different (eg } i \cdot i = -1).\]

\[\text{5I’ve adapted this example from [1].}\]
We need to consider two cases for this integral. If \( n \neq -1 \) then:

\[
ir^{n+1} \int_0^{2\pi} e^{i(n+1)t} \, dt = ir^{n+1} \left. \frac{e^{i(n+1)t}}{i(n+1)} \right|_0^{2\pi} = ir^{n+1} \left[ \frac{1}{i(n+1)} - \frac{1}{i(n+1)} \right] = 0
\]

but if \( n = -1 \), then:

\[
ir^{n+1} \int_0^{2\pi} e^{i(n+1)t} \, dt = i \int_0^{2\pi} \, dt = 2\pi i
\]

You can verify that this is the same answer as we would get using the residue theorem.

**Problem 3 - Using the inverse Laplace transform**

You can use partial fractions with an LT table to evaluate the inverse Laplace Transform or you can use the formula:

\[
g(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} G(s)e^{st} \, ds
\]

where \( G(s) \) is the Laplace Transform of \( g(t) \) and \( c \) is a real number which is greater than the real part of any point where \( G(s) \) is singular (eg if \( G(s) \) has a single pole at \( s_1 \) then \( c > \text{Re}(s_1) \)). The first thing to note is that the above integral is a line integral along a contour parallel to the imaginary axis. Thus conceptually you could attempt to evaluate it by doing the variable change \( s = ix \) and evaluating it like an improper integral on the real axis. However, you would in most cases get an integral that can’t be solved with regular integration methods. This is where contour integration comes in handy.

**Problem Statement.** Using the above formula find the inverse Laplace transform of:

\[
G(s) = \frac{e^{-2s}}{s^2(s - 1)(s^2 + 9)}
\]

**Solution.** We are solving for:

\[
g(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{-2s}}{s^2(s - 1)(s^2 + 9)} e^{st} \, ds
\]

4
where $c > 1$. The contour we are evaluating and the poles of this function are drawn in figure[4]. The trick to solving this is to notice that the following integral can be written as two pieces:

$$
\frac{1}{2\pi i} \oint_{\Gamma \pm} \frac{e^{st}}{s^2(s-1)(s^2+9)} ds = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{st}}{s^2(s-1)(s^2+9)} e^{st} ds + \frac{1}{2\pi i} \int_{C^\pm} \frac{e^{st}}{s^2(s-1)(s^2+9)} e^{st} ds
$$

Where $\Gamma \pm$ describes the two possible infinite closed half circle with flat edge parallel with the imaginary axis (see figure[4]. $C^\pm$ describes the two possible open half circles. We notice a few things: The integral we want to solve for is the second term, the third term might evaluate to zero by Jordan’s Lemma depending on which of the two half-circles we are considering, and lastly $\Gamma \pm$ will be a simple closed curve so we can apply the residue theorem. Note that considering either half-circle will not change the second term (which is the one we are solving for).

To evaluate the above integrals we consider each half circle separately. Note that when $t - 2 < 0$ we can only use Jordan’s Lemma to get rid of the third term if the contour is a half-circle closed in the right half-plane ($C^c_r$). However, if $t - 2 > 0$ then we can only use it if our half-circle is in the left half-plane ($C^c_l$).

Let’s first deal with the case $t - 2 < 0$. We close our half-circle contour in the right half plane. We can then apply Jordan’s Lemma to conclude that the third term is zero. In addition there are no poles in the right half-plane above our contour (as we have defined $c$ that way). A closed contour not containing any poles will integrate to zero. Therefore we have shown that when $t - 2 < 0$ then $g(t) = 0$.

Now we consider the case $t - 2 > 0$. We close the contour in the left half-plane and can show by Jordan’s Lemma that the third term is zero. To find the first term we apply the residue theorem:

$$
\frac{1}{2\pi i} \oint_{\Gamma} \frac{e^{st}}{s^2(s-1)(s^2+9)} ds = \text{Res}_{s=0} \left[ \frac{e^{st}}{s^2(s-1)(s^2+9)} \right] + \frac{1}{2\pi i} \int_{C^{-3i}} \frac{e^{st}}{s^2(s-1)(s^2+9)} e^{st} ds
$$

We now solve for the residues. The pole at $s=0$ is of second order and all the others are simple poles.

$$
\text{Res}_{s=0} \left[ \frac{e^{st}}{s^2(s-1)(s^2+9)} \right] = \lim_{s \to 0} \frac{d}{ds} \left[ \frac{e^{st}}{s^2(s-1)(s^2+9)} \right] = \frac{e^{st}}{9}
$$

$$
\text{Res}_{s=1} \left[ \frac{e^{st}}{s^2(s-1)(s^2+9)} \right] = \lim_{s \to 1} \frac{e^{st}}{s^2(s-1)(s^2+9)} = \frac{e^{st}}{10}
$$

$$
\text{Res}_{s=-3i} \left[ \frac{e^{st}}{s^2(s+1)(s^3+9)} \right] = \lim_{s \to -3i} \frac{e^{st}}{s^2(s-1)(s^3+9)} = \frac{1}{R} e^{i(3t-2)+\theta}
$$

where $R = \sqrt{162^2+54^2}$ and $\theta = \tan^{-1}\left(\frac{54}{162}\right)$.

Summing the residues for $t - 2 > 0$:

$$
\frac{1}{2\pi i} \oint_{c-i\infty}^{c+i\infty} \frac{e^{st}}{s^2(s-1)(s^2+9)} ds = \frac{1}{9} t - \frac{e^{t-2}}{10} + \frac{2}{R} \cos(3(t-2)+\theta)
$$

Knowing that $g(t) = 0$ when $t - 2 < 0$ we can write:

$$
g(t) = \left( \frac{1}{9} t - \frac{e^{t-2}}{10} + \frac{2}{R} \cos(3(t-2)+\theta) \right) H(t-2)
$$

where $H(t)$ is the Heaviside step function.

Other resources

http://www.youtube.com/watch?v=_3p_E9jZOU8
http://www.youtube.com/watch?v=3he49fbzYvI
http://www.damtp.cam.ac.uk/user/reh10/lectures/nst-mmii-chapter5.pdf
References