In this homework, $\mathscr{L}$ stands for the Laplace transform, $\tilde{f}(s) = \mathscr{L}[f(t)] = \int_0^\infty dt e^{-st} f(t)$.

1. Simple Laplace transforms. Provide $\tilde{f}(s)$ for the following $f(t)$.

   (a) $f(t) = 1$

   $$\tilde{f}(s) = 1/s$$

   (b) $f(t) = 0$ if $t < a$, $f(t) = 1$ if $t \geq a$, and $a \geq 0$.

   $$\tilde{f}(s) = e^{-sa}/s$$

   (c) $f(t) = e^{-at}$

   $$1/(s+a)$$

   (d) $f(t) = t$

   $$1/s^2$$

2. Laplace transforms of powers through generating functions.

   (a) Let $f(t) = \lim_{a\to 0}(-d/da)e^{-at}$. What is $f(t)$?

   $$f(t) = t$$

   (b) Let $\tilde{g}(s,a) = -(d/da)\mathscr{L}[e^{-at}]$. What is $\tilde{g}(s,a)$?

   $$\tilde{g}(s,a) = -(d/da)1/(s+a) = 1/(s+a)^2$$

   (c) If functions are well-behaved, then $\tilde{f}(s)$ should equal $\lim_{a\to 0}\tilde{g}(s,a)$. Calculate $\tilde{f}(s)$ with this method.

   $$\lim_{a\to 0} 1/(s+a)^2 = 1/s^2$$

   (d) Describe how you could use this approach to calculate $\mathscr{L}(t^n)$ for $n = 1,2,3,\ldots$.

   $$\begin{aligned} \mathscr{L}(t^n) &= \lim_{a\to 0}(-d/da)^n \mathscr{L}(e^{-at}) \\ &= \lim_{a\to 0}(-d/da)^n (s+a)^{-1} \\ &= \lim_{a\to 0} n!(s+a)^{-(n+1)} = n!/s^{n+1} \end{aligned}$$

3. Convolution. Let $\tilde{f}(s) = 1/(s+a)$ and $\tilde{g}(s) = 1/(s+b)$.

   (a) What are $f(t)$ and $g(t)$?

$$f(t) = e^{-at}$$
$$g(t) = e^{-bt}$$

   (b) What is $\int_0^\infty dt \int_0^t dt' e^{-st} f(t-t')g(t')$?

$$1/[(s+a)(s+b)]$$