The $\mathscr{L}$ operator is the Laplace transform, $\mathscr{L}[f(t)] = \tilde{f}(s) = \int_0^\infty dt e^{-st} f(t)$. The $\star$ operator is convolution, $f \star g(t) = \int_0^t dt' f(t-t') g(t')$.

1. Standard Laplace transform proofs.

   (a) Prove that $\mathscr{L}[f \star g(t)] = \tilde{f}(s)\tilde{g}(s)$.

   Start with the definitions of $\mathscr{L}$ and $\star$:

   $$\mathscr{L}[f \star g(t)] = \int_0^\infty dt e^{-st} \int_0^t dt' f(t-t') g(t').$$

   The standard approach is to multiply by 1 inside the integral, in this case $1 = e^{st'} e^{-st'}$:

   $$\ldots = \int_0^\infty dt \int_0^t dt' e^{-st} e^{st'} e^{-st'} f(t-t') g(t').$$

   Now change variables from (t, t') to $(t-t', t')$. Both limits of integration are $[0, \infty)$.

   $$\ldots = \int_0^\infty d(t-t') \int_0^\infty dt' e^{-s(t-t')} f(t-t') e^{-st'} g(t').$$

   The integrals are now independent:

   $$\ldots = \Big[\int_0^\infty d(t-t') e^{-s(t-t')} f(t-t')\Big] \Big[\int_0^\infty dt' e^{-st'} g(t')\Big] = \tilde{f}(s)\tilde{g}(s).$$

   (b) Prove that $\mathscr{L}[\dot{f}(t)] = s\tilde{f}(s) - f(0)$.

   Start with the definition of $\mathscr{L}$:

   $$\mathscr{L}[\frac{d}{dt} f(t)] = \int_0^\infty dt e^{-st} \frac{df(t)}{dt}.$$

   Integrate by parts:

   $$\begin{aligned} \ldots &= \int_0^\infty \frac{d}{dt}\big[e^{-st} f(t) dt\big] - \int_0^\infty \Big[\frac{de^{-st}}{dt}\Big] f(t) dt \\ &= e^{-st} f(t)\big|_0^\infty + s \int_0^\infty e^{-st} f(t) dt \\ &= \lim_{t\to\infty} e^{-st} f(t) - f(0) + s\tilde{f}(s). \end{aligned}$$

   The Laplace transform $\mathscr{L}[f]$ only exists if $\lim_{t\to\infty} e^{-st} f(t) = 0$. So, when $\mathscr{L}[f]$ is defined,

   $$\mathscr{L}[\dot{f}(t)] = s\tilde{f}(s) - f(0).$$

2. Whence comes that 2π? Suppose we consider eigenfunctions of the time derivative operator that are periodic on the interval $t = -T/2$ to $T/2$. The eigenfunction with eigenvalue $i\omega$ is defined $A_\omega \phi_\omega(t)$, where A_ω is a normalization constant and $\phi_\omega(t)$ is the eigenfunction at time t. The phase of the eigenfunctions are fixed by requiring that $\phi_\omega(0) = 1$.

   (a) Provide the function $\phi_\omega(t)$.

$$\frac{d}{dt} A_\omega \phi_\omega(t) = i\omega A_\omega \phi_\omega(t) \quad (1)$$

$$\phi_\omega(t) = e^{i\omega t} \quad (2)$$

   (b) The periodicity requirement is that $\phi_\omega(-T/2) = \phi_\omega(+T/2)$. What values of ω are permitted?

$$e^{-i\omega T/2} = e^{i\omega T/2} \quad (3)$$

$$1 = e^{i\omega T} \quad (4)$$

$$\omega T = 2n\pi, \quad n = 0, \pm 1, \pm 2, \ldots \quad (5)$$

$$\omega = 2n\pi/T, \quad n = 0, \pm 1, \pm 2, \ldots. \quad (6)$$

   (c) What is the spacing $\Delta\omega$ between permitted values? This quantization of permitted frequencies is the same phenomenon as the quantization of energy levels for particles in confining potentials, for example a particle in a box or an electron confined by the positive charge of a nucleus.

$$\Delta\omega = \frac{2\pi}{T} \quad (7)$$

   (d) The dot product for discrete vectors becomes an inner product for continuous functions. The inner product of functions $f(t)$ and $g(t)$ is

$$\langle f|g\rangle = \int_{-T/2}^{T/2} dt f^*(t) g(t),$$

   where $f^*(t)$ is the complex conjugate of $f(t)$. What is $\langle \phi_\omega | \phi_\omega \rangle$?

$$\langle \phi_\omega | \phi_\omega \rangle = \int_{-T/2}^{T/2} dt e^{-i\omega t} e^{i\omega t} \quad (8)$$

$$= T \quad (9)$$

(e) What is $\langle\phi_\omega|\phi_{\omega'}\rangle$ for $\omega \neq \omega'$?

$$\begin{aligned} \langle\phi_\omega|\phi'_\omega\rangle &= \int_{-T/2}^{T/2} dt e^{-i\omega t} e^{i\omega' t} && (10)\\ &= \int_{-T/2}^{T/2} dt e^{i(\omega'-\omega)t} && (11)\\ &= \frac{\exp[i(\omega'-\omega)t]}{i(\omega'-\omega)}\Big|_{-T/2}^{T/2} && (12)\\ &= 0 && (13) \end{aligned}$$

when $\omega' - \omega \neq 0$. In the limit that $\varepsilon \equiv \omega' - \omega \to 0$, L'ôpital's rule gives

$$\begin{aligned} \lim_{\varepsilon\to 0} \frac{\exp[i\varepsilon t]}{i\varepsilon}\Big|_{-T/2}^{T/2} &= \lim_{\varepsilon\to 0} \frac{(d/d\varepsilon)\exp[i\varepsilon t]}{(d/d\varepsilon)i\varepsilon}\Big|_{-T/2}^{T/2} && (14)\\ &= \lim_{\varepsilon\to 0} \frac{it\exp[i\varepsilon t]}{i}\Big|_{-T/2}^{T/2} && (15)\\ &= T. && (16) \end{aligned}$$

(f) For normalization, we require that

$$\sum_{\omega'} \Delta\omega \langle A_{\omega'}\phi_{\omega'}|A_\omega\phi_\omega\rangle = 1.$$

Note that only the term with $\omega = \omega'$ contributes to the integral. What is the resulting value for the square magnitude of the normalization constant, $|A^*_\omega A_\omega|$? How does the normalization constant depend on the frequency ω?

Since only the term with $\omega' = \omega$ contributes to the sum,

$$\begin{aligned} 1 &= \sum_{\omega'} \Delta\omega \langle A_{\omega'}\phi_{\omega'}|A_\omega\phi_\omega\rangle = \Delta\omega |A^*_\omega A_\omega| \langle\phi_\omega|\phi_\omega\rangle && (17)\\ &= \frac{2\pi}{T}|A^*_\omega A_\omega| T. && (18)\\ & && (19) \end{aligned}$$

The normalization is

$$|A^*_\omega A_\omega| = 1/2\pi,$$

and it is the same for every ω.

(g) In the limit $T \to \infty$, $\Delta\omega \to 0$, the eigenfunctions are written $A(\omega)\phi(\omega,t)$. What is $A(\omega)\phi(\omega,t)$? For historical reasons ("it seemed like a good idea at the time"), many fields (including ours) effectively use $\phi(\omega,t)$ as the eigenfunction and interpret $|A(\omega)|^2$ as a normalization for the frequency-domain integral.

$$A(\omega) = \frac{1}{\sqrt{2\pi}} \tag{20}$$

$$\phi(\omega,t) = e^{-i\omega t} \tag{21}$$

(h) Suppose a function $f(t)$ is a superposition of eigenfunctions, $f(t) = \int_{-\infty}^{\infty} d\omega \hat{f}(\omega)\phi(\omega,t)$. How do we extract $\hat{f}(\omega)$?

Begin with the definition,

$$f(t) = \int_{-\infty}^{\infty} d\omega' \hat{f}(\omega)\phi(\omega',t).$$

Since eigenfunctions are orthogonal, we can project out a specific component as an inner product:

$$\langle\phi(\omega)|f\rangle = \int_{-\infty}^{\infty} d\omega' \langle\phi(\omega)|\hat{f}(\omega')\phi(\omega')\rangle = \int_{-\infty}^{\infty} d\omega' \hat{f}(\omega')\langle\phi(\omega)|\phi(\omega')\rangle.$$

We showed above for a discrete basis that

$$\frac{1}{2\pi}\sum_{\omega'} \Delta\omega \hat{f}_{\omega'}\langle\phi_\omega|\phi_{\omega'}\rangle = \hat{f}_\omega,$$

which defines $(\Delta\omega/2\pi)\langle\phi_\omega|\phi_{\omega'}\rangle$ as the discrete (Kronecker) delta function $\delta_{\omega,\omega'}$. In the continuous limit,

$$\frac{1}{2\pi}\langle\phi(\omega)|f\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega' \hat{f}(\omega')\langle\phi(\omega)|\phi(\omega')\rangle = \hat{f}(\omega),$$

which defines the continuous (Dirac) delta function as

$$\delta(\omega-\omega') = \frac{1}{2\pi}\langle\phi_\omega|\phi_{\omega'}\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt e^{i(\omega-\omega')t}.$$

Returning to the requested problem, and using the definition of the inner product as a time integral,

$$\hat{f}(\omega) = \frac{1}{2\pi}\langle\phi(\omega)|f\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt e^{-i\omega t} f(t).$$