1. Evaluate $\oint dz \frac{1}{z}$ for the following closed contours expressed in polar coordinates, $z = re^{i\theta}$ with $r > 0$.

   (a) A single counter-clockwise loop, $\theta = 0$ to 2π.

   $$2\pi i$$

   (b) A single clockwise loop, $\theta = 2\pi$ to 0.

   $$-2\pi i$$

   (c) A double counter-clockwise loop, $\theta = 0$ to 4π.

   $$4\pi i$$

   (d) A complicated path with m counter-clockwise loops and n clockwise loops.

   $$(m-n)2\pi i$$

2. Evaluate

   $$\oint \frac{ds}{2\pi i} \frac{e^{st}}{(s+a)(s+b)}$$

   over the following closed contours, each a single counter-clockwise loop, with $a > b > 0$.

   (a) A contour including $-a$ but excluding $-b$.

   $$\oint \frac{ds}{2\pi i} \frac{e^{-at}e^{(s+a)t}}{(s+a)(s+b)} = \frac{e^{-at}}{-a+b} \oint \frac{ds}{2\pi i} \frac{1}{s+a} = \frac{e^{-at}}{b-a}$$

   (b) A contour including $-b$ but excluding $-a$.

   $$\oint \frac{ds}{2\pi i} \frac{e^{-bt}e^{(s+b)t}}{(s+a)(s+b)} = \frac{e^{-bt}}{-b+a} \oint \frac{ds}{2\pi i} \frac{1}{s+b} = \frac{e^{-bt}}{a-b}$$

   (c) A contour up the imaginary axis and then closed in the left half-plane: $z = -i\infty \to +i\infty \to -\infty + i\infty \to -\infty - i\infty \to -i\infty$.

   $$\frac{e^{-at} - e^{-bt}}{b-a}$$

(d) A contour up the imaginary axis and then closed in the right half-plane: $z = -i\infty \to +i\infty \to \infty + i\infty \to \infty - i\infty \to -i\infty$.

$$0$$

(e) Not a closed contour but just a path up the imaginary axis, $z = -i\infty \to +i\infty$.

$$\frac{e^{-at} - e^{-bt}}{b-a}$$

3. Consider the integral

$$\int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} \frac{e^{st}}{(s+a)^2}$$

with $a > 0$.

(a) Evaluate the integral using contour integrals by expanding e^{st} in a Taylor series around $s = -a$, $e^{st} = e^{-at} e^{(s+a)t} = e^{-at}[1 + (s+a)t + (s+a)^2 t^2/2 + (s+a)^3 t^3/3! + \ldots]$.

$$\oint \frac{ds}{2\pi i} \frac{e^{-at}(s+a)t}{(s+a)^2} = te^{-at}$$

(b) Evaluate the integral by first evaluating

$$\int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} \frac{e^{st}}{(s+a)(s+b)}$$

and then taking the limit $b \to a$.

$$\lim_{b \to a} \frac{e^{-at} - e^{-bt}}{b-a} = \lim_{b \to a} \frac{te^{-bt}}{1} = te^{-at}$$

(c) Evaluate

$$\int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} \frac{e^{st}}{s^2}.$$

$$t$$

4. Contour integrals simplify Fourier transforms. Let

$$f(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{i\omega t}}{\omega^2 + a^2}$$

with $a > 0$.

(a) Where are the poles?

$$\omega = \pm ia$$

(b) For $t > 0$, what is the value of the integrand for $\omega \to +i\infty$ and for $\omega \to -i\infty$? Can you close the contour in the upper half-plane or the lower half-plane?

$$\lim_{\omega \to +i\infty} \frac{e^{i\omega t}}{\omega^2 + a^2} = 0$$

$$\lim_{\omega \to -i\infty} \frac{e^{i\omega t}}{\omega^2 + a^2} = -\infty$$

Close contour in upper half-plane.

(c) What is $f(t)$ for $t > 0$?

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-at} e^{i(\omega - ia)t}}{(\omega + ia)(\omega - ia)} = \frac{e^{-at}}{2ia} \oint \frac{d\omega}{2\pi} \frac{1}{\omega - ia} = \frac{e^{-at}}{2a}$$

(d) For $t < 0$, in which half-plane should the contour be closed? What is $f(t)$ for $t < 0$? Close in lower half-plane.

$$\frac{e^{at}}{-2ia} \oint \frac{d\omega}{2\pi} \frac{1}{\omega + ia} = \frac{e^{at}}{-4ia\pi}(0 - 2\pi i) = \frac{e^{at}}{2a}$$

Note that loop is clockwise.

(e) Noting that $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$, what is

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\cos(\omega t)}{\omega^2 + a^2}?$$

What is

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\sin(\omega t)}{\omega^2 + a^2}?$$

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\cos(\omega t)}{\omega^2 + a^2} = Re \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{i\omega t}}{\omega^2 + a^2} = \frac{e^{-a|t|}}{2a}$$

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\cos(\omega t)}{\omega^2 + a^2} = Im \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{i\omega t}}{\omega^2 + a^2} = 0$$